Hawking Temperature of a Schwarzschild Black Hole

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What we will need

Before embarking on the investigation of the simplest possible kind of black hole, we need some simple notions of:

- general relativity;
- quantum mechanics;
- statistical mechanics.

The Jebsen-Birkhoff theorem

Two important results of Newtonian gravity:

- At any point outside a spherical mass distribution, the grav. field Φ depends only on the mass interior to that point.
- Even if the mass interior is moving spherically symmetrically, the grav. field Φ outside is constant in time.

A corresponding result in GR is the Jebsen-Birkhoff theorem (1921,1923):

- The *only* vacuum, spherically symmetric gravitational field is static.

$\Rightarrow$ The *vacuum* space-time external to a spherical mass of radius R is necessarily the one described by the Schwarzschild metric (1916).

The Schwarzschild metric

The Schwarzschild (or Schwarzschild-Droste) metric is:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \frac{1}{\left(1 - \frac{2GM}{c^2 r}\right)} dr^2 + r^2 d\Omega^2 \quad (r > R)$$

where

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta \, d\phi^2$$

- The coordinates t, r, θ, and ϕ are the coordinates used by an observer at rest a great ($\simeq \infty$) distance from the origin.
- For convenience, let us define the quantity

$$r_S \equiv \frac{2GM}{c^2}$$

as the Schwarzschild radius.

Real or apparent singularity?

From now on we will adopt units in which $c = 1$. $\left(\Rightarrow r_S \equiv 2GM\right)$

- Look at the Schwarzschild metric $\left(v^2 \equiv r_S/r\right)$:

$$ds^2 = \underbrace{-\left(1 - v^2\right)}_{g_{tt}} dt^2 + \underbrace{\frac{1}{\left(1 - v^2\right)}}_{g_{rr}} dr^2 + r^2 d\Omega^2$$

- It seems that something worrying happens at $r = r_S$:
  - g_{tt} vanishes and g_{rr} blows up.
  - And vice versa for $g^{tt} = 1/g_{tt}$ and $g^{rr} = 1/g_{rr}$.

  Is the Schwarzschild radius a physical singularity or not?
- There are many ways to see that. For example, let us make a change of coordinates (we are always free to do so).

Painlevé-Gullstrand coordinates (1921)

$$ds^2 = -(1-v^2)dt^2 + \frac{1}{(1-v^2)}dr^2 + r^2 d\Omega^2 \qquad v^2 \equiv \frac{r_S}{r}$$

- Let $dt = dT - h(r)dr$. Then the metric reads

$$ds^2 = -(1-v^2)\, dT^2 + 2h(1-v^2)\, dT dr + \left[-(1-v^2)h^2 + \frac{1}{1-v^2}\right] dr^2 + r^2 d\Omega^2$$

- Choose $h(r)$ s.t. the coefficient of dr^2 equals 1:

$$\left[-(1-v^2)h^2 + \frac{1}{1-v^2}\right] = 1 \qquad \Rightarrow \qquad h = \frac{v}{1-v^2}$$

- With this choice of $h(r)$, the metric becomes

$$ds^2 = -dT^2 + \left(dr + \sqrt{\frac{r_s}{r}}dT\right)^2 + r^2 d\Omega^2$$

$\Rightarrow$ In the $(T,\, r,\, \theta,\, \phi)$ coordinates, nothing blows up at $r = r_S$.

$\Rightarrow$ In contrast, $r = 0$ is a real singularity (cloaked by the event horizon).

A way to define a black hole

Consider a photon ($ds = 0$) moving along a certain radial direction toward us (fixed θ, ϕ). The rate at which the radial coordinate of the photon changes (i.e. the *apparent* speed of light) is:

$$ds^2 = 0 = -\left(1 - \frac{r_S}{r}\right)dt^2 + \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 + \cancel{r^2 d\Omega^2}$$

$$\Rightarrow v_r \equiv \frac{dr}{dt} = \left(1 - \frac{r_S}{r}\right)$$

Note that as $r \to r_S$, $v_r \to 0$ (light is frozen in time).
A photon at $r = r_S$ will never reach us.

The spherical surface at $r = r_S$ is called event horizon.

A star (say, a massive one: $M \gtrsim 10 M_\odot$) which collapses down within its Schwarzschild radius is called a black hole.

Is a black hole really “black”?

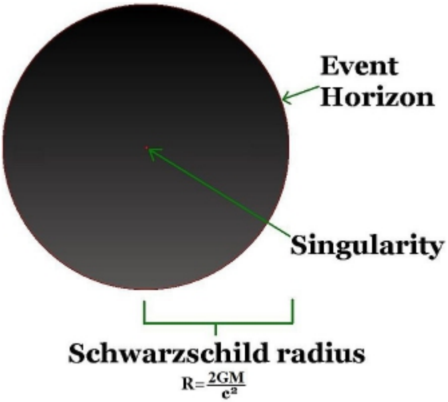


Image from *Wikipedia*

To answer this question we will make use of a nice correspondence between quantum mechanics and statistical mechanics...

Wisdom from Quantum Mechanics

In quantum mechanics, given a Hamiltonian $\hat{H}$ and a quantum state which we want to evolve after time T, what we need is:

- Initial state $|I\rangle$
- Final state $|F\rangle$
- Evolution operator $e^{-\frac{i\hat{H}T}{\hbar}}$

$\Rightarrow$ Probability amplitude: $\mathcal{Z} = \langle F| \, e^{-\frac{i\hat{H}T}{\hbar}} \, |I\rangle$

Wisdom from Statistical Mechanics

Consider a system with a Hamiltonian s.t.

$$\hat{H}\,|\lambda\rangle = E_\lambda\,|\lambda\rangle$$

The probability of the system being in some particular state $|\lambda\rangle$ with energy E_λ at a temperature T is given by the Gibbs distribution $(\beta \equiv 1/k_B T)$

$$p_\lambda = \frac{e^{-\beta E_\lambda}}{Z}$$

where Z is the partition function of the system defined as a sum over states:

$$Z \equiv \sum_\lambda e^{-\beta E_\lambda} = \sum_\lambda \langle\lambda|\, e^{-\beta \hat{H}}\,|\lambda\rangle = \mathrm{Tr}\left[e^{-\beta \hat{H}}\right]$$

Any resemblance?

Look at these two formulas:

$$\mathcal{Z} = \langle F| \, e^{-\frac{i\hat{H}T}{\hbar}} \, |I\rangle$$

$$Z = \sum_{\lambda} \langle \lambda| \, e^{-\beta \hat{H}} \, |\lambda\rangle$$

Let us naively ask: is there any correspondence between $\mathcal{Z}$ and Z ?

To have $\mathcal{Z} \leftrightarrow Z$, we may perform the following computational trick:

- $T \to -i\beta\hbar$ (make time imaginary)
- $|I\rangle = |F\rangle = |\lambda\rangle$ (force every state to go back to itself)
- sum over $|\lambda\rangle$ (do it for any state)

$\Rightarrow$ Time becomes somehow cyclic.

$\Rightarrow$ β is equal to this recurrence period.

Back to the Schwarzschild spacetime

$$ds^2 = -\left(\frac{r - r_S}{r}\right)dt^2 + \left(\frac{r}{r - r_S}\right)dr^2 + r^2 d\Omega^2$$

- Consider a particle (or a photon) propagating near outside the horizon ($r \simeq r_S + \epsilon$):

$$ds^2 \simeq -\left(\frac{r - r_S}{r_S}\right)dt^2 + \left(\frac{r_S}{r - r_S}\right)dr^2 + r_S^2 d\Omega^2$$

  Now we will make three simple changes of variables.

- (1) Change the "radial" coordinate from r to ρ:

$$\rho^2 = 4r_S(r - r_S)$$

$$\rho\, d\rho = 2r_S\, dr$$

$$(r - r_S)\, d\rho^2 = r_S\, dr^2$$

- The spacetime near the horizon becomes:

$$ds^2 = -\frac{\rho^2}{4r_S^2}\, dt^2 + d\rho^2 + r_S^2\, d\Omega^2$$

Move to Euclidean space

$$ds^2 = -\frac{\rho^2}{4r_S^2}\,dt^2 + d\rho^2 + r_S^2\,d\Omega^2$$

- (2) Switch to "imaginary" time:

$$t = -it_E \qquad dt = -i dt_E$$

$$\Rightarrow \quad ds^2 = \frac{\rho^2}{4r_S^2}\,dt_E^2 + d\rho^2 + r_S^2\,d\Omega^2$$

- (3) Rescale the variable t_E:

$$t_E = 2r_S\psi \qquad dt_E = 2r_S\,d\psi$$

Thus, the metric near the horizon reads:

$$ds^2 = \underbrace{d\rho^2 + \rho^2\,d\psi^2}_{\substack{\text{2d plane metric:} \\ \text{radius } \rho \in [0,\infty] \\ \text{angle } \psi \in [0,2\pi]}} + r_S^2\,d\Omega^2$$

Hawking Temperature

$$ds^2 = d\rho^2 + \rho^2 d\psi^2 + r_S^2\, d\Omega^2$$

$\Rightarrow$ We got a cyclic time t_E with a recurrence period of 2π:

$$t = -it_E = -i2r_S\psi \quad \longrightarrow \quad -i2r_S(2\pi) = -i\,4\pi r_S$$

- In other words, we got a periodicity:

$$t \to -i(4\pi r_S)$$

- But at the very beginning the correspondence $\mathcal{Z} \leftrightarrow Z$ gave us

$$T \to -i\beta\hbar$$

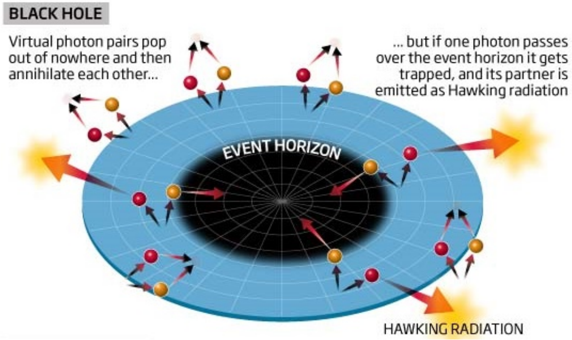
- We are tempted to identify $\beta\hbar = 4\pi r_S$:

$$\Rightarrow \quad \frac{k_B T_H}{\hbar} = \frac{1}{4\pi r_S} = \frac{1}{4\pi(2GM)} = \frac{1}{8\pi GM}$$

Restoring the proper units of c:

$$\boxed{T_H = \frac{\hbar c^3}{8\pi k_B GM}}$$

A simple heuristic physical interpretation



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Black-hole evaporation

$$r_S = \frac{2GM}{c^2} \qquad T_H = \frac{\hbar c^3}{8\pi k_B GM}$$

- The black holes of classical general relativity last forever.
- However, QFT (in curved spacetime) calculations show that the outgoing particles have a thermal black-body spectrum.
- Since a black hole radiates energy by Hawking radiation, energy conservation implies that it will lose mass.
  As the black hole becomes lighter it becomes hotter.
- Let us ignore all other processes other than the Hawking radiation:

$$\underbrace{\frac{dM}{dt}c^2}_{\frac{[E]}{[T]}} = -\underbrace{4\pi r_S^2}_{[L]^2}\;\underbrace{\sigma T_H^4}_{\frac{[E]}{[L]^2[T]}}$$

$$\Rightarrow t_{ev} \propto M^3$$

$\Rightarrow$ Small black holes evaporate much faster than very massive ones.

References

(A **1942** - Ω **2018**)

Image from *NASA*

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