# High-velocity regime in Kelvin wakes

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## First year PhD Seminar

Università di Pisa

# OUTLINE



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- This result is **independent** of the **shape** and **velocity** of the object.
- This is the angle formed by **ducks** in a (deep-enough) pond, **moving boats** far from the coast etc.

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- The equations governing the dynamics are:

$$\partial_t \rho + (\boldsymbol{v}\boldsymbol{\nabla})\rho = -\rho(\boldsymbol{\nabla}\cdot\boldsymbol{v})$$

**Conservation of mass**

$$\partial_t \rho\boldsymbol{v} + \boldsymbol{\nabla}(\boldsymbol{v}\rho\boldsymbol{v}) = -\boldsymbol{\nabla}P + \boldsymbol{f}$$

**Conservation of momentum**

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- And assuming stationarity (identical to the Mach cone):

$$U \cos\theta(k) = c_\varphi(k) = \sqrt{g/k}.$$

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- This implies: $\lambda(\theta) = \frac{2\pi U^2 \cos^2 \theta}{g}$

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- Thus waves are bounded within the envelope defined by: $\left.\frac{y}{x}\right|_{Max} = 2^{-3/2}$
- Which corresponds to an angle of $\sim$19.5 deg

# SOME (GENERAL?) INTUITION

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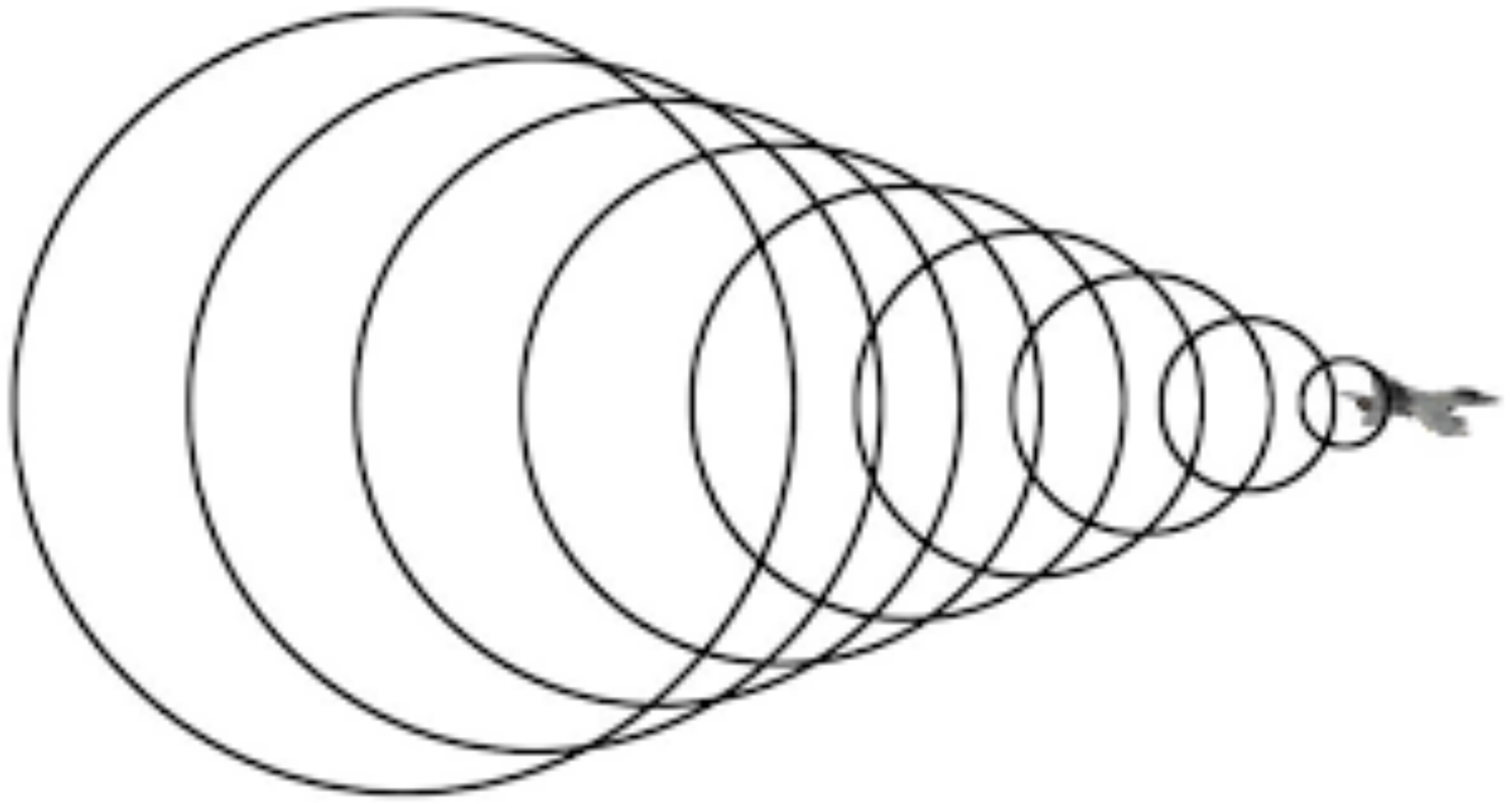
- But **supersonic physics** (together with relativity, electrodynamics,…) give us a different intuition, i.e. the **angle** should **shrink** for **high** boat **speed**.



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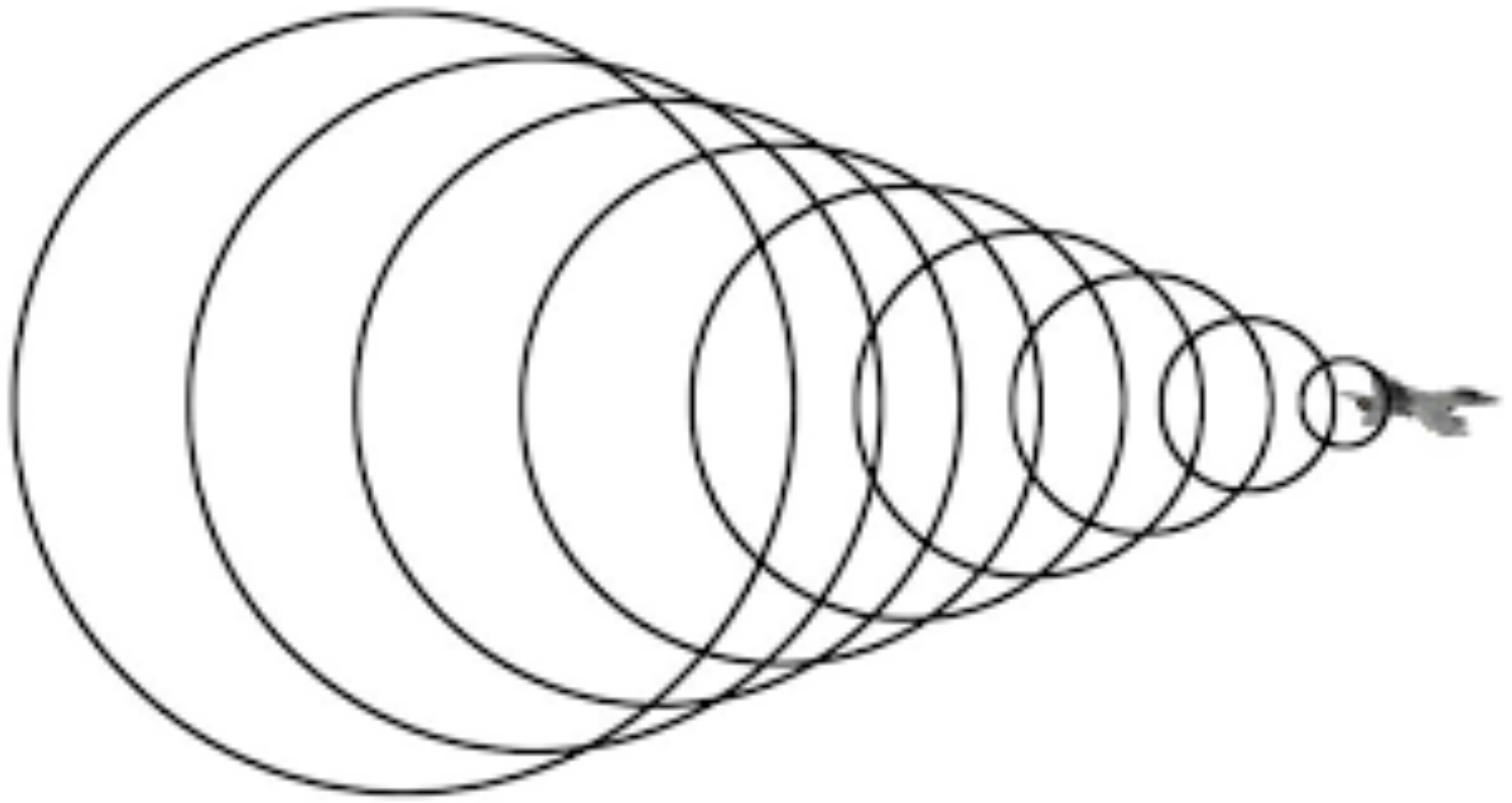

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- Is this intuition correct? (the regime is very different, here we are in the presence of an incompressible fluid with gravity as restoring force and a **non-linear dispersion relation**)

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- In absence of dedicated funding **Rabaud and Moisy** relied on the Google Earth (GE) database.
- Using GE calibration the **wake angle**, together with **size** and **speed** of the ship can be determined (after correcting for parallax effects).
- The **speed** come from the wavelength measurement and $\lambda(\theta) = \frac{2\pi U^2 \cos^2\theta}{g}$ : main source of error.

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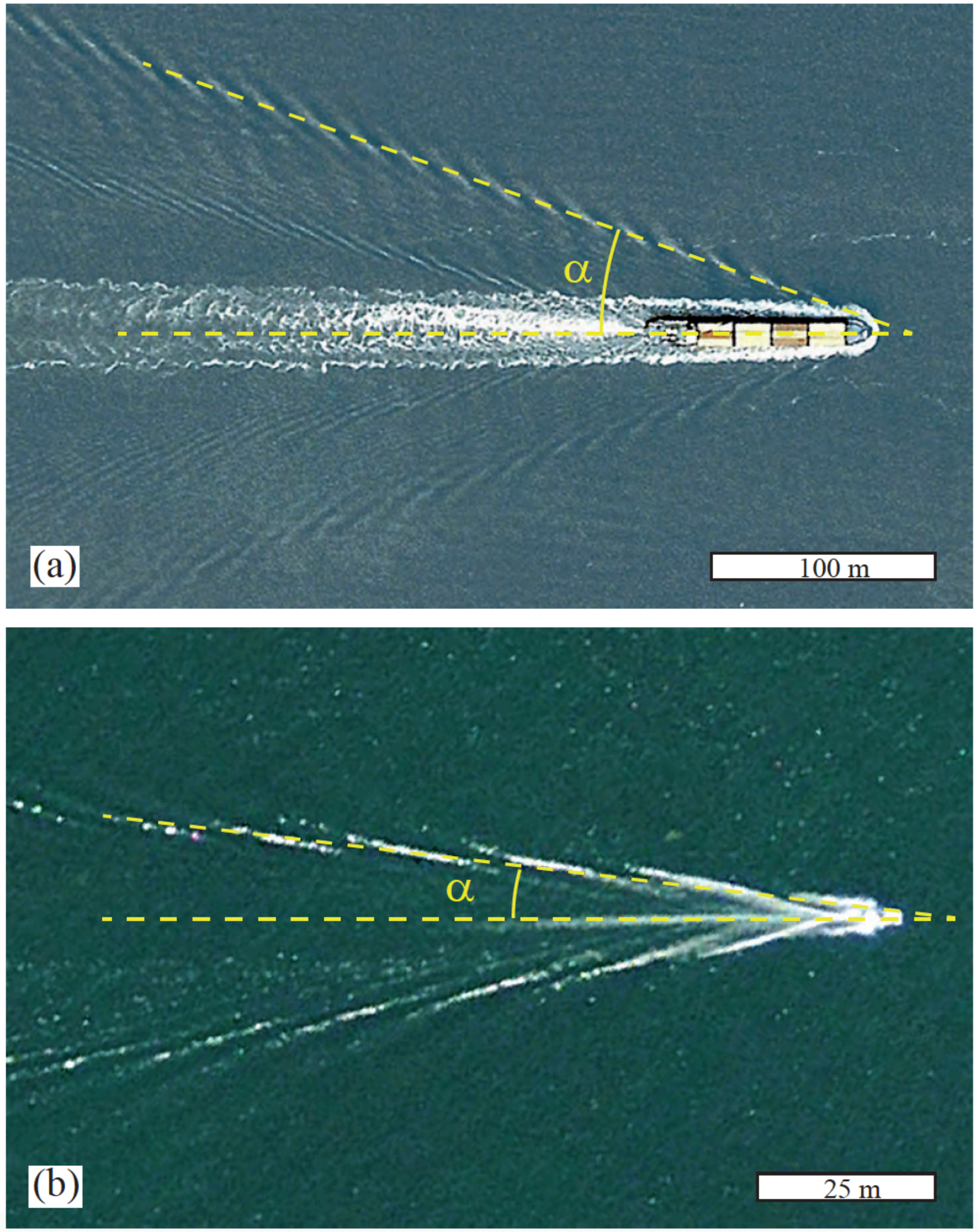




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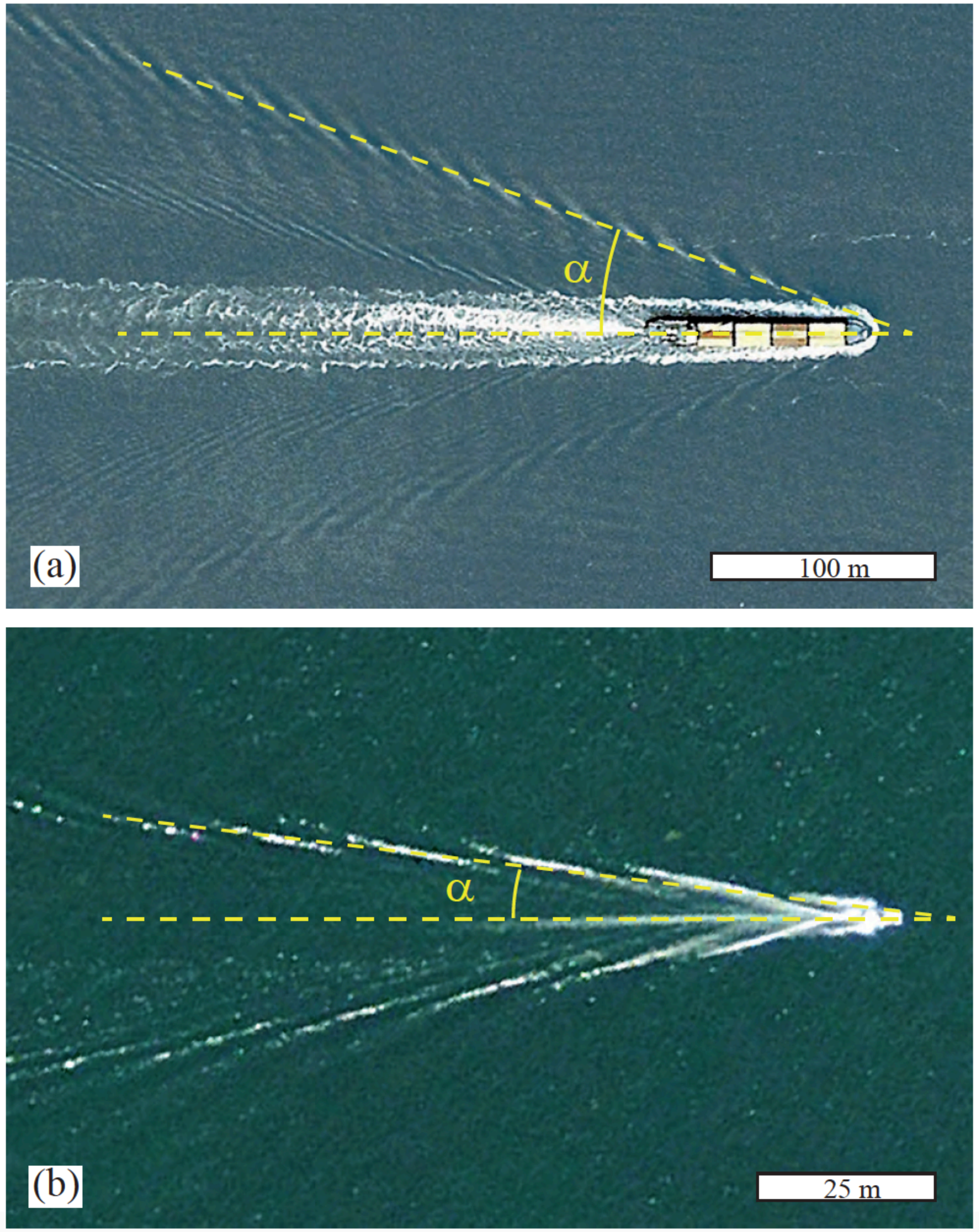
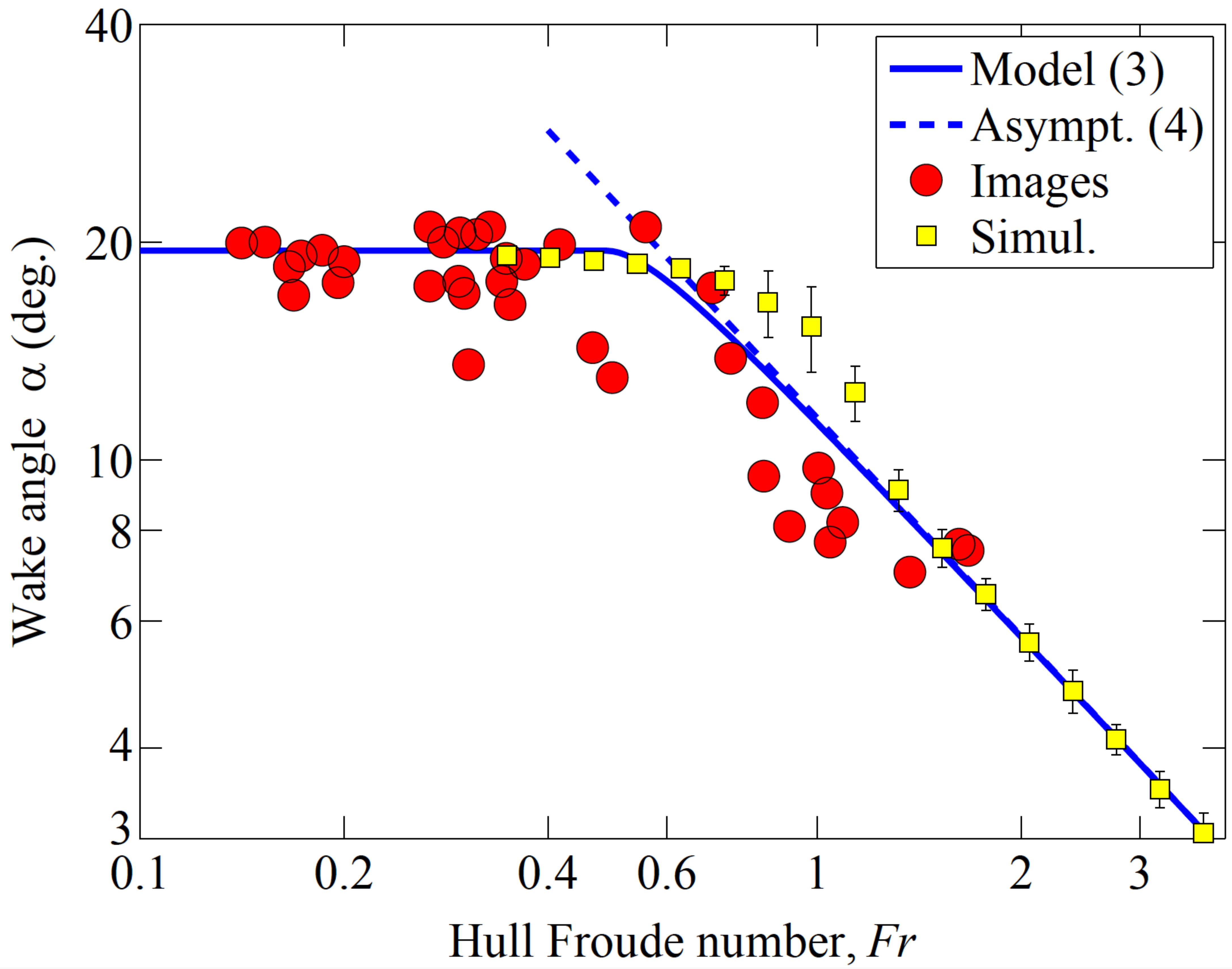



**Clear transition at large velocities!**




$$Fr = U/\sqrt{gL}$$



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- Rabaud and Moisy proposed a model for which a boat of size L cannot excite wavelength $\lambda >> L$.
- According to this model a **cutoff** must be imposed on the previous maximization.

# TRANSITION TO HIGH-VELOCITY REGIME



- This cutoff leads to:

$$\alpha = \tan^{-1}(1/\sqrt{8}) \simeq 19.47^{\circ}, \qquad Fr \leq Fr_c$$

$$\alpha = \tan^{-1} \frac{\sqrt{2\pi Fr^2 - 1}}{4\pi Fr^2 - 1}, \qquad Fr \geq Fr_c,$$

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- And in the high-velocity regime to:

$$\alpha \approx \frac{1}{2\sqrt{2\pi} Fr}$$

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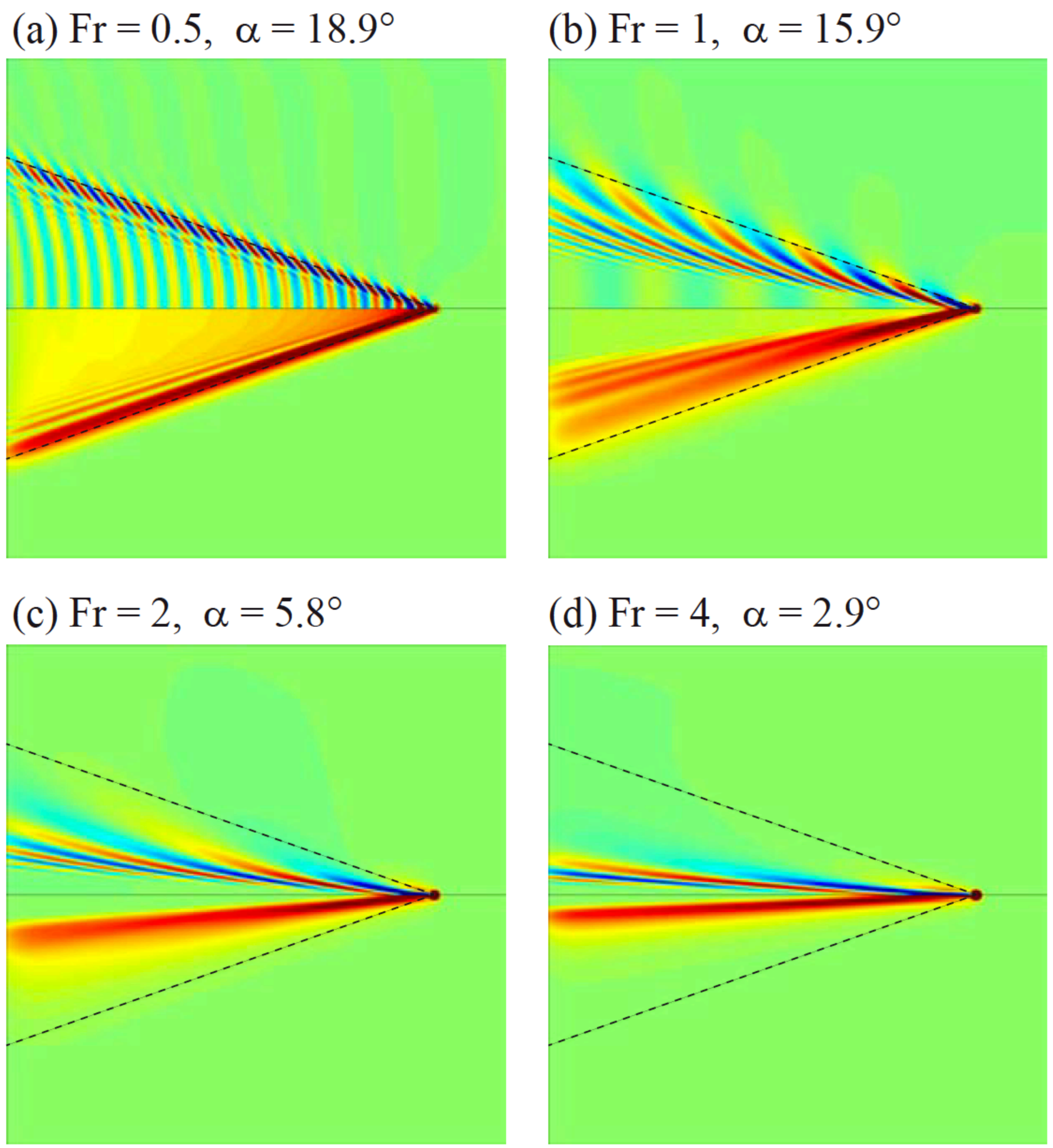

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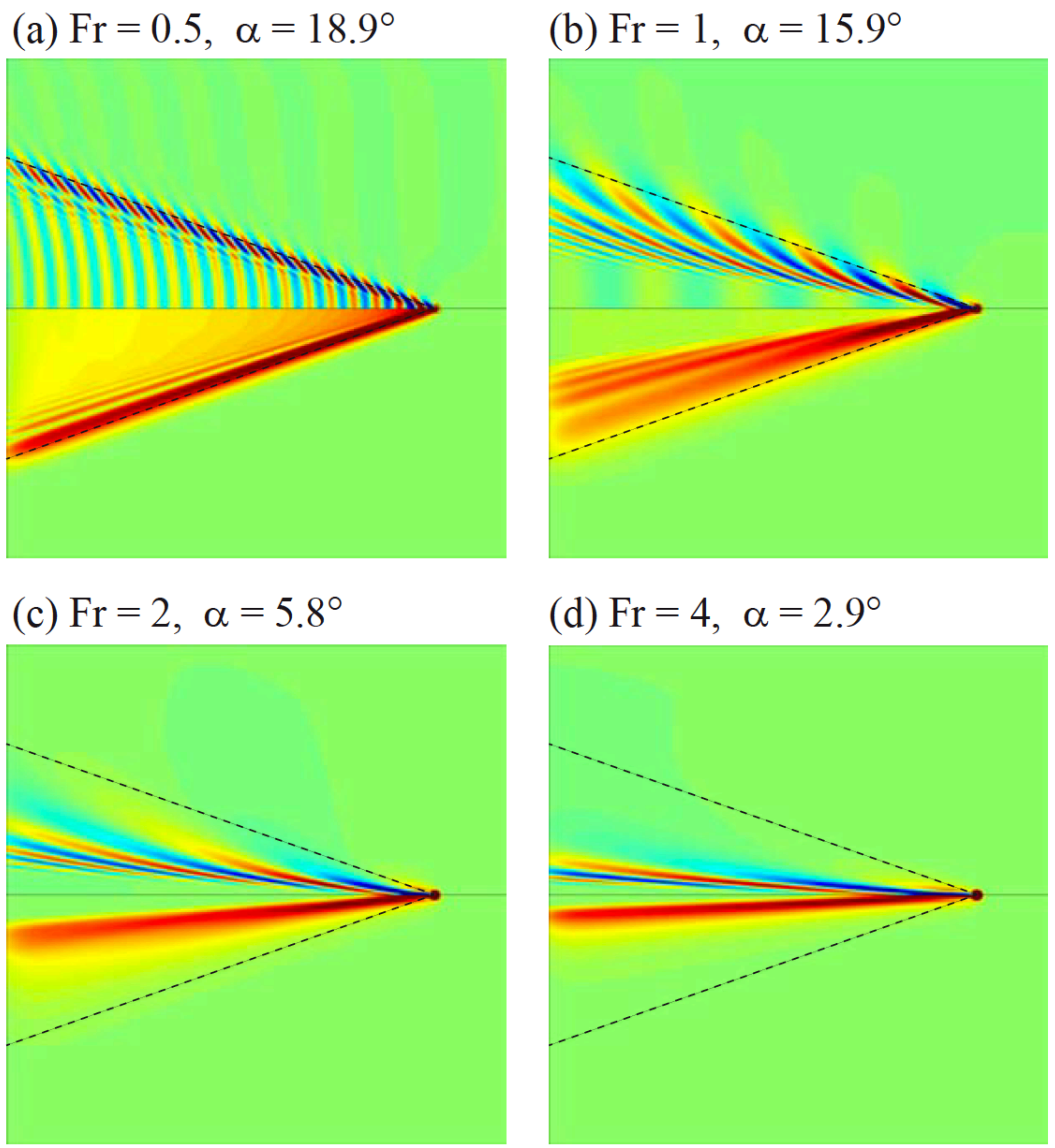
- In full agreement with the presence of both a **Kelvin-like** and a **Mach-like regime**.





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# TRANSITION TO HIGH-VELOCITY REGIME

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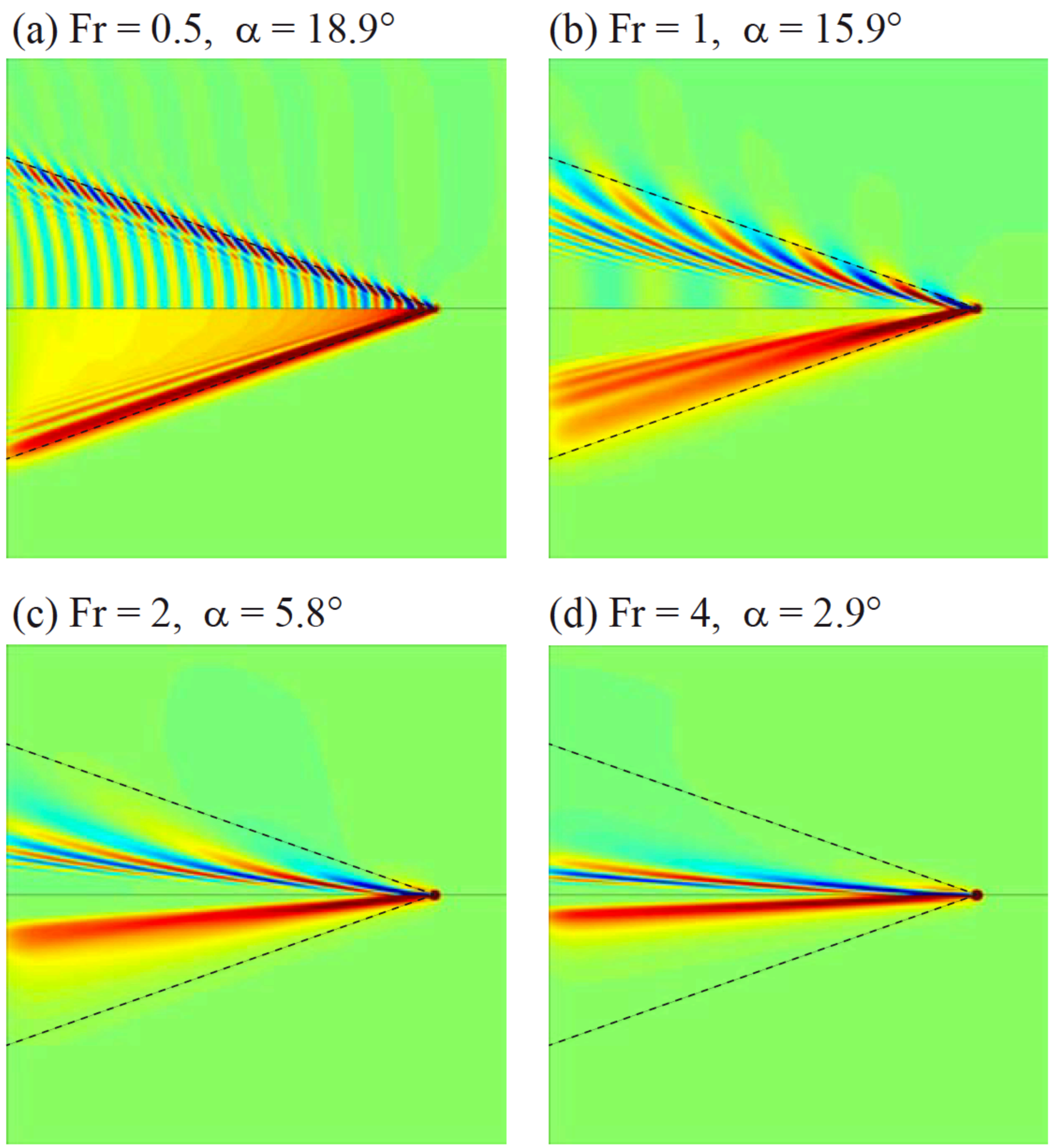
- Simulations performed using a moving gaussian pressure distribution show:
- In full agreement with the presence of both a **Kelvin-like** and a **Mach-like regime**.
- **Supersonic behaviour** reproduced from a pure **dispersive effect**!



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- Why the need to make the assumption that no $\lambda >> L$ can be excited? (large-lambda corresponding to low-energy waves should easily be excited)
- Darmon+ proved that there is **no need for such assumption**.
- They reproduced the $\frac{1}{U}$ behaviour through a **pure analytic treatment**:

$$\zeta(x,y) = -\lim_{\varepsilon\to 0} \iint \frac{\mathrm{d}k\,\mathrm{d}\theta}{4\pi^2\rho} \frac{\hat{p}(k,\theta)\, e^{-ik(\cos\theta\, x - \sin\theta\, y)}}{c(k)^2 - V^2\cos^2\theta + 2i\varepsilon V\cos\theta/k}$$

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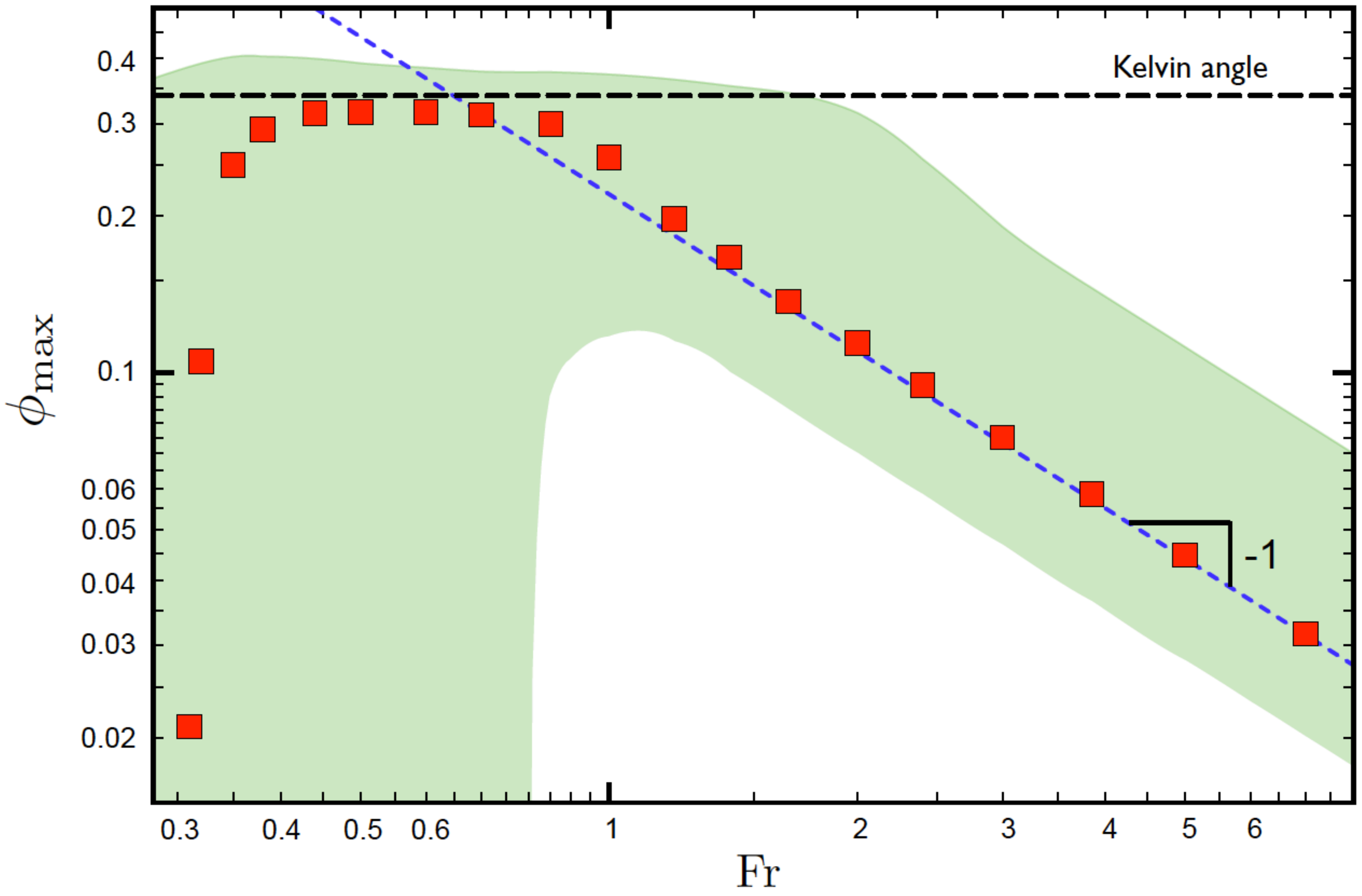

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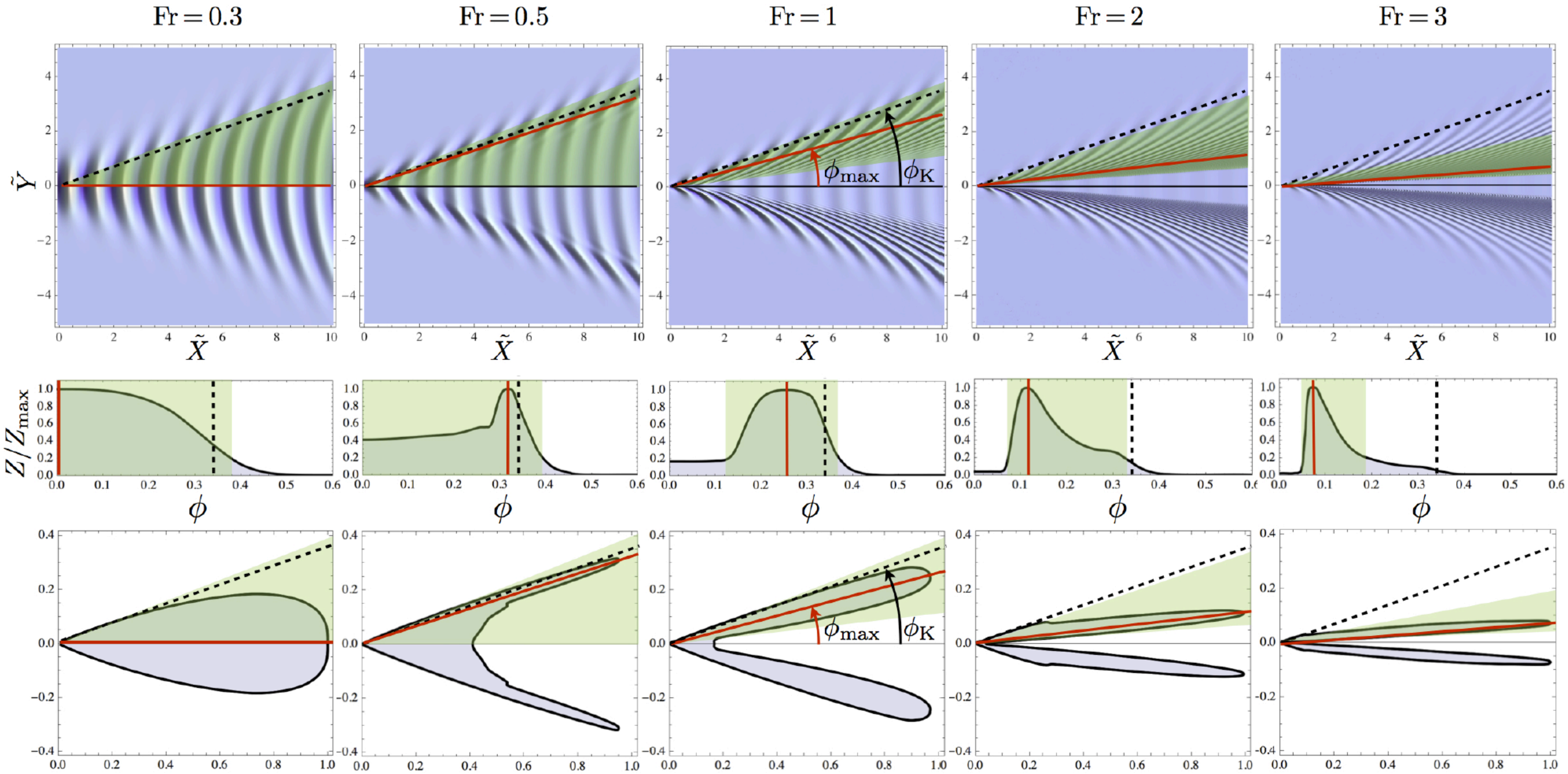

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- Why then the **Kelvin prediction** fails? (Spoiler: It doesn't…)
- The wake envelope is **always bounded** from above by **Kelvin angle**, but the waves corresponding to such angle are **not** the one with the **largest amplitude**!
- The angle corresponding to the largest amplitude instead follows the U^-1 prediction (**Supersonic behaviour** reproduced from a pure dispersive effect!)

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- We discussed the **corrections** to the constant **Kelvin angle** in the high velocity regime
- At **low velocities** (compared to a typical velocity introduced by gravity and finite-size effects) the standard result (**19.5 deg**) is recovered
- At **high velocity** a **Mach-like regime** kicks in and a $\frac{1}{U}$ behaviour is recovered
- Nonetheless the Kelvin angle prediction has been show to be robust even in the high velocity regime, although not relative to the waves at peak amplitude

# That's all Folks!