Some Lessons from Higher Spins

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- *D.Francia, AS, hep-th/0601199 (short review)*
- *D. Francia, J. Mourad, AS, hep-th/0701163, NPB to appear*

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Why Higher Spins ?

- **Key (old) problem in Field Theory**
  - Systematic treatment for important sub-class: $\varphi_{\mu_1 \ldots \mu_s} \ , \psi_{\mu_1 \ldots \mu_s}$

- **Key role in String Theory:**
  - (Non) Planar duality of tree amplitudes
  - Modular invariance and soft U.V.
  - Open-closed duality
  - Microscopic entropy counts

What we know

- *Flat-space formulation for a large (but not exhaustive) class of HS fields*

  (with a number of surprises which I will try to illustrate)

- *Extension to (A) dS backgrounds*

- *Inconsistency of more general backgrounds for individual HS fields*

  (Aragone-Deser problem)

- *Two well-defined frameworks with infinitely many interacting HS fields :*

  - *STRING THEORY: broken HS symmetries, same scale in masses and interactions*

  - *VASILIEV' EQUATIONS: unbroken HS symmetries, same scale in s=2 C.C. and interactions*
    [BACKGROUND INDEPENDENT , non Lagrangian]

Plan

- *Free fields :*
  - *"Constrained " vs "Unconstrained " Higher Spins*
  - *Non-local (geometric) & Local (compensator) forms*
  - *Relation with String Theory*

- *External currents :*
  - *A subtlety with the non-local "Einstein" tensor*
  - *An interesting by-product*

Free HS fields

- *Simplest case where structure appears* → *s=2 (linearized Einstein) :*

$$\Box h_{\mu\nu} \;-\; (\partial_\mu \partial \cdot h_\nu + \partial_\nu \partial \cdot h_\mu) \;+\; \partial_\mu \partial_\nu h' \;=\; 0$$
$$\delta h_{\mu\nu} \;=\; \partial_\mu \xi_\nu \;+\; \partial_\nu \xi_\mu$$

- *Generalize to :*

$$h_{\mu\nu} \longrightarrow \varphi_{\mu_1 \ldots \mu_s}$$
$$\Lambda_\mu \longrightarrow \Lambda_{\mu_1 \ldots \mu_{s-1}}$$

→ φ, Λ

- *Fronsdal (1978):*

$$\mathcal{F}_{\mu_1 \ldots \mu_s} \equiv \Box \varphi_{\mu_1 \ldots \mu_s} \;-\; (\partial_{\mu_1} \partial \cdot \varphi_{\mu_2 \ldots \mu_s} + \ldots) \;+\; \left(\partial_{\mu_1} \partial_{\mu_2} \varphi'_{\mu_3 \ldots \mu_s} + \ldots\right) \;=\; 0$$
$$\delta \varphi_{\mu_1 \ldots \mu_s} \;=\; \partial_{\mu_1} \Lambda_{\mu_2 \ldots \mu_s} \;+\; \ldots$$

$$\mathcal{F} \equiv \Box \varphi \;-\; \partial \partial \cdot \varphi \;+\; \partial^2 \varphi' \;=\; 0$$
$$\delta \varphi \;=\; \partial \Lambda$$

But . . .

The Fronsdal constraints

- *For s>2 gauge invariance of field equation ONLY if :*
- *For s>3 gauge invariance of Lagrangian ONLY if :*

$$\Lambda' = 0$$
$$\varphi'' = 0$$

$$\delta \mathcal{F} = 3\,\partial^3 \Lambda'$$
$$\partial \cdot \mathcal{F} - \frac{1}{2}\,\partial \mathcal{F}' = -\,\frac{3}{2}\,\partial^3 \varphi''$$

- *"Unconstrained" gauge symmetry?*
  - *Non-local extension of field equations (and actions) (Francia, AS, 2002…)*
  - *Spin-(s-3) α and spin-(s-4) β (Francia, Tsulaia, AS, 2003, 2005)*
  - *BRST-like with O(s) extra fields (Pashnev, Tsulaia, 1998…)*
  - *[BRST-like with 5 extra fields] (Buchbinder, Krykhtin, Reshetnyak, 2007)*

Non-local equations

- *Spin s=3 :*

$$\mathcal{F}_{\mu\nu\rho} \equiv \Box\varphi_{\mu\nu\rho} - (\partial_\mu \partial\cdot\varphi_{\nu\rho} + \ldots) + \left(\partial_\mu\partial_\nu\varphi'_\rho + \ldots\right)$$

$$\mathcal{F}_{\mu\nu\rho} = 0$$
$$\delta\,\mathcal{F}_{\mu\nu\rho} = 3\,\partial_\mu\partial_\nu\partial_\rho\,\Lambda'$$

$\Rightarrow$

$$\mathcal{F}_{\mu\nu\rho} - \frac{1}{3}\,\frac{1}{\Box}\left(\partial_\mu\partial_\nu\mathcal{F}'_\rho + \ldots\right) = 0$$
$$\mathcal{F}_{\mu\nu\rho} - \frac{\partial_\mu\partial_\nu\partial_\rho}{\Box^2}\,\partial\cdot\mathcal{F}' = 0$$

- *Spin s > 3 :*

$$\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + \frac{1}{(n+1)(2n+1)}\,\frac{\partial^{\,2}}{\Box}\,\mathcal{F}^{(n)\,\prime} - \frac{1}{n+1}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{F}^{(n)}$$
$$\delta\mathcal{F}^{(n)} = (2n+1)\frac{\partial^{2n+1}}{\Box^{n-1}}\Lambda^{[n]}$$

  - *proceed up to gauge invariance*

- *result can take a suggestive form :*

$$\mathcal{F} = 3\,\partial^{\,3}\,\mathcal{H}$$
$$\delta\mathcal{H} = \Lambda'$$

Local compensator equations

- *Alternatively* LOCAl equation with spin-(s-3) *compensator* :
- Spin s=3 : (ant. by Schwinger !) $\mathcal{F}_{\mu\nu\rho} \;=\; 3\partial_\mu\partial_\nu\partial_\rho\alpha$
- In general :

$$\mathcal{F} \;=\; 3\,\partial^{\,3}\,\alpha$$
$$\varphi'' \;=\; 4\partial\cdot\alpha \;+\; \partial\alpha'$$
$$\delta\alpha \;=\; \Lambda'$$

- Second reflects Bianchi identity
- Results extend nicely to **(A)dS backgrounds**

What do we gain with an unconstrained gauge symmetry?

Glimpses of HS geometry

- *HS connections and curvatures:* *(de Wit and Freedman, 1980)*

- *s=2:* $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \rightarrow \Gamma \sim \partial h \ , \ \ R \sim \partial^2 h$

- *s > 2:* $\varphi_{\mu_1 \ldots \mu_s} \rightarrow \Gamma \sim \partial^{\,s-1} \varphi \ , \ \ R \sim \partial^{\,s} \varphi$

$$\mathcal{R}_{\mu_1 \ldots \mu_s ; \nu_1 \ldots \nu_s}$$

- *s = 2n :* $\frac{1}{\Box^{n-1}} \ \mathcal{R}^{[n]}{}_{;\nu_1 \ldots \nu_s} \ = \ 0$
- *s = 2n+1 :* $\frac{1}{\Box^{n}} \ \partial_\mu \ \mathcal{R}^{\mu[n]}{}_{;\nu_1 \ldots \nu_s} \ = \ 0$

$$R_{\mu\nu} = 0$$
$$\partial_\mu F^{\mu\nu} = 0$$

[Mixed symmetry: Bekaert, Boulanger, de Medeiros, Hull, 2003, …]

Minimal local Lagrangians

(Francia, AS, 2005; Francia, Mourad and AS, 2007)

- "Minimal" local Lagrangians with **unconstrained** gauge symmetry:

$$\left[\mathcal{F} = \Box\varphi - \partial\partial\cdot\varphi + \partial^2\varphi'\right]$$
$$\mathcal{A} = \mathcal{F} - 3\,\partial^{\,3}\,\alpha$$
$$\partial\cdot\mathcal{A} - \frac{1}{2}\,\partial\mathcal{A}' = -\frac{3}{2}\,\partial^3\left(\varphi'' - 4\partial\cdot\alpha - \partial\alpha'\right)$$

- The Lagrangians are:

$$\mathcal{L} = \frac{1}{2}\,\varphi\left(\mathcal{A} - \frac{1}{2}\,\eta\,\mathcal{A}'\right) - \frac{3}{4}\binom{s}{3}\alpha\,\partial\cdot\mathcal{A}' + 3\binom{s}{4}\beta\left[\varphi'' - 4\,\partial\cdot\alpha - \partial\alpha'\right]$$

Can be nicely extended to **(A)dS backgrounds**

Fermions

Fermions :

- *Recall spin-3/2 :*

$$\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho = 0 \rightarrow \gamma^{\nu\rho}\partial_\nu\psi_\rho = 0$$

$$\not\partial\psi_\mu - \partial_\mu \not\psi = 0$$

$$\delta\psi_\mu = \partial_\mu \epsilon$$

- *Spin – (n+1/2) :*

$$\not\partial\psi_{\mu_1\ldots\mu_n} - \left(\partial_{\mu_1} \not\psi_{\mu_2\ldots\mu_n} + \ldots\right) = 0$$

$$\delta\psi_{\nu_1\ldots\mu_n} = \partial_{\mu_1} \epsilon_{\mu_2\ldots\mu_n} + \ldots$$

$$\not\epsilon = 0 \, , \quad \not\psi\,' = 0$$

Local compensator eqs :

(Francia, AS, Tsulaia, 2003)

$$\not\partial\psi - \partial \not\psi = -2\partial^2\xi$$

$$\delta\psi = \partial\epsilon \, , \quad \delta\xi = \not\epsilon$$

$$\not\psi\,' = 2\partial\cdot\xi + \partial\xi\,' + \not\partial \not\xi$$

String Theory, triplets and HS

- *String Field Theory:*

$$\mathcal{Q}\,|\Psi\rangle = 0$$
$$\delta\,|\Psi\rangle = \mathcal{Q}\,|\Lambda\rangle$$

(Kato and Ogawa, 1982; Witten; Neveu, West et al, 1985)

NO trace constraints !

- $\alpha' \to \infty$ limit (symmetric tensors): TRIPLETS
  - propagate s,s-2,s-4, ….

$$\Box\,\varphi = \partial\,C\,,$$
$$\partial\cdot\varphi - \partial\,D = C$$
$$\Box\,D = \partial\cdot C$$

(A. Bengtsson, 1986, Henneaux and Teitelboim, 1986; Pashnev and Tsulaia, 1998; Francia, Tsulaia, AS, 2003)

$$\delta\varphi = \partial\Lambda\,,$$
$$\delta C = \Box\,\Lambda\,,$$
$$\delta D = \partial\cdot\Lambda$$

(On shell) truncation:

$$\varphi\,' - 2\,D = \partial\alpha$$
$$(\delta\alpha = \Lambda\,')$$

$$\mathcal{F} = 3\,\partial^3\,\alpha$$
$$\varphi\,'' = 4\,\partial\cdot\alpha + \partial\alpha\,'$$

→ Elim. $\alpha \to$ N. Loc. Geom. Eqs.

Off-Shell truncation of triplets

Off-shell reduction of triplets :

(Buchbinder, Krykhtin, Reshetnyak 2007)

- *start from a triplet (s,s-2,…)*
- *add two (gauge invariant) Lagrange multipliers*
- *Lagrangian :*

$$\Box\,\varphi \;=\; \partial\, C\;,\qquad \partial\cdot\varphi \;-\; \partial\, D \;=\; C\;,\qquad \Box\, D \;=\; \partial\cdot C$$

$$\lambda\;:\quad \varphi\,' - 2\,D \;-\; \partial\alpha \;=\; 0$$
$$\mu\;:\quad D\,' \;-\; \partial\cdot\alpha \;=\; 0$$

$$\begin{aligned}\mathcal{L} \;=\; & -\frac{1}{2}\,(\partial_\mu\varphi)^2 \;+\; s\,\partial\cdot\varphi\, C \;+\; s(s-1)\,\partial\cdot C\, D \\ & +\; \frac{s(s-1)}{2}\,(\partial_\mu D)^2 \;-\; \frac{s}{2}\,C^{\,2} \\ & +\; \lambda\Big(\varphi\,' \;-\; 2\,D \;-\; \partial\alpha\Big) \;+\; \mu\Big(D\,' \;-\; \partial\cdot\alpha\Big)\end{aligned}$$

λ and μ : set to zero by the field equations

External currents

- Residues of current exchanges reflect the degrees of freedom

- For s=1 :

$$p^2 A_\mu - p_\mu \, p \cdot A = J_\mu$$

$$p^2 J^\mu A_\mu = J^\mu J_\mu \;\Rightarrow\; J_i J_i$$

- For all s :

$$\mathcal{A} - \frac{1}{2}\eta \mathcal{A}' + \eta^2 \mathcal{B} = J$$

$$\partial \cdot \mathcal{A}' - (2\partial + \eta\,\partial\cdot)\mathcal{B} = 0$$

$$\varphi'' - 4\,\partial \cdot \alpha - \partial \alpha' = 0$$

External currents : local case

$$\varphi'' - 4\,\partial\cdot\alpha - \partial\alpha' = 0$$

- $\mathcal{K}$ "doubly traceless" using double trace constraint
- $\mathcal{B}$: determines multiplier β for double trace constraint

$$\mathcal{A} - \frac{1}{2}\eta\,\mathcal{A}' = J - \eta^2\,\mathcal{B} \equiv \mathcal{K}$$
$$\varphi'' - 4\,\partial\cdot\alpha - \partial\,\alpha' = 0$$

$$Traceless\ \ Proj\ :\ \sum_0^N \rho_n(D,s)\,\eta^n\,V^{[n]}$$
$$\rho_{n+1}(D,s) = -\frac{\rho_n(D,s)}{D+2(s-n-2)}$$

$$\mathcal{K} = J + \sum_{n=2}^N \sigma_n\,\eta^n\,J^{[n]}$$
$$\sigma_n + [D+2(s-n-3)]\,\{2\sigma_{n+1} + [D+2(s-n-4)]\,\sigma_{n+2}\} = 0$$
$$\sigma_n = (-n+1)\,\rho_n(D-2,s)$$

$$\mathcal{A} = \mathcal{K} + \rho_1(D-2,s)\,\eta\,\mathcal{K}' = \sum_n \rho_n(D-2,s)\,\eta^n\,J^{[n]}$$
$$\sum_{n=0}^N \rho_n(D-2,s)\,\frac{s!}{n!\,(s-2n)!\,2^n}\,J^{[n]}\cdot J^{[n]}$$

The exchange involves, correctly, a traceless conserved current

External currents : non-local case

How about the non-local version of the theory?

Apparently: different choices for the field equation, EQUIVALENT without currents

S=3 :

$$\mathcal{F}_{\mu\nu\rho} - \frac{1}{3}\frac{1}{\Box}\left(\partial_\mu\partial_\nu\mathcal{F}'_\rho + \ldots\right) = 0$$

$$\mathcal{F}_{\mu\nu\rho} - \frac{\partial_\mu\partial_\nu\partial_\rho}{\Box^2}\,\partial\cdot\mathcal{F}' = 0$$

$$\Rightarrow \quad \frac{1}{\Box}\,\eta_{\alpha\beta}\,\partial_\mu\,\mathcal{R}^{\mu\alpha\beta}{}_{;\nu_1\nu_2\nu_3} = 0$$

$$\mathcal{F} = 3\,\partial^3\alpha \quad \Rightarrow \quad \mathcal{F}^{(n)} = (2n+1)\,\frac{\partial^{2n+1}}{\Box^{n-1}}\,\alpha^{[n-1]}$$

$$\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + \frac{1}{(n+1)(2n+1)}\frac{\partial^2}{\Box}\mathcal{F}^{(n)\,\prime} - \frac{1}{n+1}\frac{\partial}{\Box}\,\partial\cdot\mathcal{F}^{(n)}$$

$$\delta\mathcal{F}^{(n)} = (2n+1)\frac{\partial^{2n+1}}{\Box^{n-1}}\Lambda^{[n]}$$

$$\partial\cdot\mathcal{F}^{(n)} - \frac{1}{2n}\,\partial\mathcal{F}^{(n)\,\prime} = -\left(1+\frac{1}{2n}\right)\frac{\partial^{2n+1}}{\Box^{n-1}}\,\varphi^{(n+1)}$$

Bianchi identity: changes after every iteration

External currents : non-local case

Naively: $\mathcal{G}^{(n)} \equiv \sum_{p=0}^{n} (-1)^p \frac{(n-p)!}{2^p\, n!}\, \eta^p\, \mathcal{F}^{(n)\,[p]} = \mathcal{J}$ → $\mathcal{F}^{(n)} = \sum_{n=0}^{n} \rho_n(D-2n, s)\, \eta^n\, \mathcal{J}^{[n]}$

Incorrect current exchange !

Solution: *modify the non-local Lagrangian equation*

$$\mathcal{A}_{nl} = \mathcal{F} - 3\,\partial^3 \alpha_{nl} \;;\quad \mathcal{C}_{nl} \equiv \varphi'' - 4\,\partial\cdot\alpha_{nl} - \partial\,\alpha'_{nl} = 0$$

$$\partial\cdot\mathcal{A}_{nl} - \frac{1}{2}\,\partial\mathcal{A}'_{nl} = 0 \;\;(\text{Mod } \mathcal{C}_{nl}) \;;\quad \partial\cdot\mathcal{A}_{nl} = 2\,\mathcal{D}_{nl}$$

$$\mathcal{G}_{nl} = \mathcal{A}_{nl} - \frac{1}{2}\,\eta\,\mathcal{A}'_{nl} + \eta^2\,\mathcal{D}_{nl} + \ldots + \eta^{n+2}\,\frac{\mathcal{D}_{nl}^{[n]}}{2^n\, n!} \qquad (s = 2n+4 \text{ or } s = 2n+5)$$

$$\mathcal{A}_{nl} = \sum_{k=0}^{n+1} a_k\, \frac{\partial^{2k}}{\Box^k}\, \mathcal{F}^{(n+1)\,[k]} \;;\quad a_k = (-1)^{k+1}\,(2k-1)\,\frac{n+2}{n-1}\prod_{j=-1}^{k-1}\frac{n+j}{n-j+1}$$

$$\mathcal{D}_{nl} = \frac{1}{2}\sum_{k=2}^{n+1} a_k \left\{ \frac{1}{2k-3}\,\frac{\partial^{2(k-2)}}{\Box^{k-2}}\,\mathcal{F}^{(n+1)\,[k]} + \frac{2n+4k+1}{2(2k-1)(n-k+1)}\,\frac{\partial^{2(k-1)}}{\Box^{k-1}}\,\mathcal{F}^{(n+1)\,[k+1]} + \frac{n+k+1}{2(n-k)(n-k+1)}\,\frac{\partial^{2k}}{\Box^k}\,\mathcal{F}^{(n+1)\,[k+2]} \right\}$$

For instance :

$$\mathcal{A}_3 = \frac{1}{\Box}\,\partial\cdot\mathcal{R}' + \frac{\partial^2}{2\,\Box^2}\,\partial\cdot\mathcal{R}''$$

$$\mathcal{A}_4 = \frac{1}{\Box}\,\mathcal{R}'' + \frac{1}{2}\,\frac{\partial^2}{\Box^2}\,\mathcal{R}''' - 3\,\frac{\partial^4}{\Box^3}\,\mathcal{R}^{[4]}$$

$$\mathcal{A}_5 = \frac{1}{\Box^2}\,\partial\cdot\mathcal{R}'' + \frac{2}{3}\,\frac{\partial^2}{\Box^3}\,\partial\cdot\mathcal{R}''' - 3\,\frac{\partial^4}{\Box^4}\,\partial\cdot\mathcal{R}^{[4]}$$

$$\mathcal{D}_4 = -\frac{3}{8}\,\frac{1}{\Box}\,\mathcal{R}^{[4]}$$

$$\mathcal{D}_5 = -\frac{5}{8\,\Box^2}\,\partial\cdot\mathcal{R}^{[4]}$$

VD-V-Z Discontinuity for HS

J ———— J

$m = 0 \; : \quad T_{\mu\nu} T^{\mu\nu} \; - \; \frac{1}{2} (T\,')^2$ (van Dam, Veltman; Zakharov, 1970)

$m \neq 0 \; : \quad T_{\mu\nu} T^{\mu\nu} \; - \; \frac{1}{3} (T\,')^2$

For all s and D, m=0 :

$$\sum_{n=0}^{N} \rho_n(D-2,s) \, \frac{s!}{n!\,(s-2n)!\,2^n} \, J^{[n]} \cdot J^{[n]}$$

$$\rho_{n+1}(D,s) \; = \; - \, \frac{\rho_n(D,s)}{D + 2(s-n-2)}$$

$\Rightarrow$ $s = 2 \; : \quad T_{\mu\nu} T^{\mu\nu} \; - \; \frac{1}{D-2} (T\,')^2$

- VDVZ discontinuity follows in general comparing D and (D+1) massless exchanges
- First present for s=2 via D-dependence of $\rho(D,s)$
- For all s: can describe irreducibly a massive field a' la Scherk-Schwarz from (D+1) dimensions :

  [e.g. for s=2 : $h_{MN} \rightarrow (h_{\mu\nu} \cos(my), A_\mu \sin(my), \varphi \cos(my))$]

$$\sum_{n=0}^{N} \rho_n(D-2,s) \, \frac{s!}{n!\,(s-2n)!\,2^n} \, J^{[n]} \cdot J^{[n]} \quad \Longleftrightarrow \quad \sum_{n=0}^{N} \rho_n(D-1,s) \, \frac{s!}{n!\,(s-2n)!\,2^n} \, J^{[n]} \cdot J^{[n]}$$

(A)dS extension, first discussed, for s=2, by Higuchi and Porrati

Discontinuity → smooth interpolation in $(mL)^2$

Outlook

- *Precise link with String Theory :*
    - *Vasiliev equations : effective theory for first open Regge trajectory ?*
- *Systematics of HS currents and interactions :*
    - *Where does String Theory lie within HS gauge theory ?*
- *HS geometry (vs algebraic structure as in Vasiliev equations) :*
    - *Key lesson for String Theory : background independence*