

# *Some Lessons from Higher Spins*

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- *D. Francia, AS, hep-th/0601199 (short review)*
- *D. Francia, J. Mourad, AS, hep-th/0701163, NPB to appear*



*Pisa/Florence, March 2007*

# Why Higher Spins ?

- *Key (old) problem in Field Theory*

- *Systematic treatment for important sub-class:*

$$\varphi_{\mu_1 \dots \mu_s} , \psi_{\mu_1 \dots \mu_s}$$

- *Key role in String Theory:*

- *(Non) Planar duality of tree amplitudes*
- *Modular invariance and soft  $\mathcal{U}, \mathcal{V}$ .*
- *Open-closed duality*
- *Microscopic entropy counts*



# What we know

- *Flat-space formulation for a large (but not exhaustive) class of HS fields*

*(with a number of surprises which I will try to illustrate )*

- *Extension to (A) dS backgrounds*

- *Inconsistency of more general backgrounds for individual HS fields*

*(Aragone-Deser problem)*

- *Two well-defined frameworks with infinitely many interacting HS fields :*

- *STRING THEORY: broken HS symmetries, same scale in masses and interactions*
- *VASILIEV' EQUATIONS: unbroken HS symmetries, same scale in  $s=2$  C.C. and interactions*  
*[BACKGROUND INDEPENDENT, non Lagrangian]*

# Plan

- *Free fields :*

- *“Constrained ” vs “Unconstrained “ Higher Spins*
- *Non-local (geometric) & Local (compensator) forms*
- *Relation with String Theory*

- *External currents :*

- *A subtlety with the non-local “Einstein” tensor*
- *An interesting by-product*



# Free HS fields

- *Simplest case* where structure appears  $\rightarrow s=2$  (linearized Einstein):

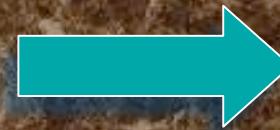
$$\square h_{\mu\nu} - (\partial_\mu \partial \cdot h_\nu + \partial_\nu \partial \cdot h_\mu) + \partial_\mu \partial_\nu h' = 0$$

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

- Generalize to:

$$h_{\mu\nu} \rightarrow \varphi_{\mu_1 \dots \mu_s}$$

$$\Lambda_\mu \rightarrow \Lambda_{\mu_1 \dots \mu_{s-1}}$$



$$\varphi$$

$$\Lambda$$

- *Fronsdal (1978):*

$$\mathcal{F}_{\mu_1 \dots \mu_s} \equiv \square \varphi_{\mu_1 \dots \mu_s} - (\partial_{\mu_1} \partial \cdot \varphi_{\mu_2 \dots \mu_s} + \dots) + (\partial_{\mu_1} \partial_{\mu_2} \varphi'_{\mu_3 \dots \mu_s} + \dots) = 0$$

$$\delta \varphi_{\mu_1 \dots \mu_s} = \partial_{\mu_1} \Lambda_{\mu_2 \dots \mu_s} + \dots$$

$$\mathcal{F} \equiv \square \varphi - \partial \partial \cdot \varphi + \partial^2 \varphi' = 0$$

$$\delta \varphi = \partial \Lambda$$

*But ...*

# The Fronsdaal constraints

- For  $s > 2$  gauge invariance of field equation *ONLY* if:
- For  $s > 3$  gauge invariance of Lagrangian *ONLY* if:

$$\begin{aligned}\Lambda' &= 0 \\ \varphi'' &= 0\end{aligned}$$

$$\begin{aligned}\delta \mathcal{F} &= 3 \partial^3 \Lambda' \\ \partial \cdot \mathcal{F} - \frac{1}{2} \partial \mathcal{F}' &= -\frac{3}{2} \partial^3 \varphi''\end{aligned}$$

- “Unconstrained” gauge symmetry?
  - *Non-local extension* of field equations (and actions) (Francia, AS, 2002...)
  - Spin- $(s-3)$   $\alpha$  and spin- $(s-4)$   $\beta$  (Francia, Tsulaia, AS, 2003, 2005)
  - *BRST-like* with  $O(s)$  extra fields (Pashnev, Tsulaia, 1998...)
  - [*BRST-like* with 5 extra fields] (Buchbinder, Krykhtin, Reshetnyak, 2007)



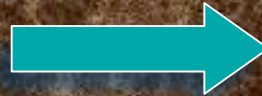
# Non-local equations

■ *Spin  $s=3$ :*

$$\mathcal{F}_{\mu\nu\rho} \equiv \square \varphi_{\mu\nu\rho} - (\partial_\mu \partial \cdot \varphi_{\nu\rho} + \dots) + (\partial_\mu \partial_\nu \varphi'_\rho + \dots)$$

$$\mathcal{F}_{\mu\nu\rho} = 0$$

$$\delta \mathcal{F}_{\mu\nu\rho} = 3 \partial_\mu \partial_\nu \partial_\rho \Lambda'$$



$$\begin{aligned} \mathcal{F}_{\mu\nu\rho} - \frac{1}{3} \frac{1}{\square} (\partial_\mu \partial_\nu \mathcal{F}'_\rho + \dots) &= 0 \\ \mathcal{F}_{\mu\nu\rho} - \frac{\partial_\mu \partial_\nu \partial_\rho}{\square^2} \partial \cdot \mathcal{F}' &= 0 \end{aligned}$$

■ *Spin  $s > 3$ :*

$$\begin{aligned} \mathcal{F}^{(n+1)} &= \mathcal{F}^{(n)} + \frac{1}{(n+1)(2n+1)} \frac{\partial^2}{\square} \mathcal{F}^{(n)} - \frac{1}{n+1} \frac{\partial}{\square} \partial \cdot \mathcal{F}^{(n)} \\ \delta \mathcal{F}^{(n)} &= (2n+1) \frac{\partial^{2n+1}}{\square^{n-1}} \Lambda^{[n]} \end{aligned}$$

■ *proceed up to gauge invariance*

■ *result can take a suggestive form:*

$$\begin{aligned} \mathcal{F} &= 3 \partial^3 \mathcal{H} \\ \delta \mathcal{H} &= \Lambda' \end{aligned}$$

# Local compensator equations

- *Alternatively* LOCAL equation with spin-(s-3) *compensator*:

- Spin  $s=3$  : (ant. by Schwinger ! )  $\mathcal{F}_{\mu\nu\rho} = 3\partial_\mu\partial_\nu\partial_\rho\alpha$

- In general:

$$\begin{aligned}\mathcal{F} &= 3\partial^3\alpha \\ \varphi'' &= 4\partial\cdot\alpha + \partial\alpha' \\ \delta\alpha &= \Lambda'\end{aligned}$$

- Second reflects Bianchi identity
- Results extend nicely to (A)dS backgrounds

*What do we gain with an unconstrained gauge symmetry?*



# Glimpses of HS geometry

- *HS connections and curvatures:*

(de Wit and Freedman, 1980)

- $s=2$ :  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \rightarrow \Gamma \sim \partial h, R \sim \partial^2 h$

- $s > 2$ :  $\varphi_{\mu_1 \dots \mu_s} \rightarrow \Gamma \sim \partial^{s-1} \varphi, R \sim \partial^s \varphi$

$$\mathcal{R}_{\mu_1 \dots \mu_s; \nu_1 \dots \nu_s}$$

- $s = 2n$ :  $\frac{1}{\square^{n-1}} \mathcal{R}^{[n]}_{;\nu_1 \dots \nu_s} = 0$

- $s = 2n+1$ :  $\frac{1}{\square^n} \partial_\mu \mathcal{R}^{\mu[n]}_{;\nu_1 \dots \nu_s} = 0$

$$R_{\mu\nu} = 0$$

$$\partial_\mu F^{\mu\nu} = 0$$

[Mixed symmetry: Bekeert, Boulanger, de Medeiros, Hull, 2003, ...]

# Minimal local Lagrangians

(Francia, AS, 2005; Francia, Mourad and AS, 2007)

- “Minimal” local Lagrangians with **unconstrained** gauge symmetry:

$$[\mathcal{F} = \square\varphi - \partial\partial\cdot\varphi + \partial^2\varphi']$$

$$\mathcal{A} = \mathcal{F} - 3\partial^3\alpha$$

$$\partial\cdot\mathcal{A} - \frac{1}{2}\partial\mathcal{A}' = -\frac{3}{2}\partial^3(\varphi'' - 4\partial\cdot\alpha - \partial\alpha')$$

- The Lagrangians are:

$$\mathcal{L} = \frac{1}{2}\varphi\left(\mathcal{A} - \frac{1}{2}\eta\mathcal{A}'\right) - \frac{3}{4}\binom{s}{3}\alpha\partial\cdot\mathcal{A}' + 3\binom{s}{4}\beta\left[\varphi'' - 4\partial\cdot\alpha - \partial\alpha'\right]$$

Can be nicely extended to **(A)dS backgrounds**



# Fermions

## Fermions :

▪ Recall spin-3/2 :

$$\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho = 0 \rightarrow \gamma^{\nu\rho}\partial_\nu\psi_\rho = 0$$

$$\not{\partial}\psi_\mu - \partial_\mu\psi = 0$$

$$\delta\psi_\mu = \partial_\mu\epsilon$$

▪ Spin - (n+1/2) :

$$\not{\partial}\psi_{\mu_1\dots\mu_n} - (\partial_{\mu_1}\psi_{\mu_2\dots\mu_n} + \dots) = 0$$

$$\delta\psi_{\nu_1\dots\mu_n} = \partial_{\mu_1}\epsilon_{\mu_2\dots\mu_n} + \dots$$

$$\not{\epsilon} = 0, \quad \psi' = 0$$

*Local compensator eqs :*

(Francia, AS, Tsulaia, 2003)

$$\not{\partial}\psi - \partial\psi = -2\not{\partial}^2\xi$$

$$\delta\psi = \partial\epsilon, \quad \delta\xi = \not{\epsilon}$$

$$\psi' = 2\not{\partial}\cdot\xi + \partial\xi' + \not{\partial}\not{\xi}$$

# String Theory, triplets and HS

## ■ String Field Theory:

$$\begin{aligned} Q |\Psi\rangle &= 0 \\ \delta |\Psi\rangle &= Q |\Lambda\rangle \end{aligned}$$

**NO trace constraints !**

(Kato and Ogawa, 1982; Witten; Neveu, West et al, 1985)

## ■ $\alpha' \rightarrow \infty$ limit (symmetric tensors): **TRIPLETS**

- propagate  $s, s-2, s-4, \dots$

(A. Bengtsson, 1986, Henneaux and Teitelboim, 1986; Pashnev and Tsulaia, 1998; Francia, Tsulaia, AS, 2003)

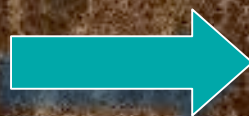
$$\begin{aligned} \square \varphi &= \partial C, \\ \partial \cdot \varphi - \partial D &= C \\ \square D &= \partial \cdot C \end{aligned}$$

(On shell) truncation:

$$\begin{aligned} \varphi' - 2D &= \partial \alpha \\ (\delta \alpha &= \Lambda') \end{aligned}$$

$$\begin{aligned} \delta \varphi &= \partial \Lambda, \\ \delta C &= \square \Lambda, \\ \delta D &= \partial \cdot \Lambda \end{aligned}$$

$$\begin{aligned} \mathcal{F} &= 3 \partial^3 \alpha \\ \varphi'' &= 4 \partial \cdot \alpha + \partial \alpha' \end{aligned}$$



Elim.  $\alpha \rightarrow$  N. Loc. Geom. Eqs.



# Off-Shell truncation of triplets

## Off-shell reduction of triplets :

(Buchbinder, Krykhtin, Reshetnyak 2007)

- start from a *triplet*  $(s, s-2, \dots)$
- add *two* (gauge invariant) Lagrange multipliers
- *Lagrangian* :

$$\begin{aligned} \square \varphi &= \partial C, \\ \partial \cdot \varphi - \partial D &= C \\ \square D &= \partial \cdot C \end{aligned}$$

$$\begin{aligned} \lambda : \quad \varphi' - 2D - \partial \alpha &= 0 \\ \mu : \quad D' - \partial \cdot \alpha &= 0 \end{aligned}$$

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} (\partial_\mu \varphi)^2 + s \partial \cdot \varphi C + s(s-1) \partial \cdot C D \\ & + \frac{s(s-1)}{2} (\partial_\mu D)^2 - \frac{s}{2} C^2 \\ & + \lambda (\varphi' - 2D - \partial \alpha) + \mu (D' - \partial \cdot \alpha) \end{aligned}$$

$\lambda$  and  $\mu$  : set to zero by the field equations

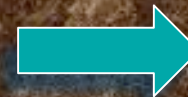
# External currents



- Residues of current exchanges reflect the *degrees of freedom*

- For  $s=1$  :  $p^2 A_\mu - p_\mu p \cdot A = J_\mu$

$$p^2 J^\mu A_\mu = J^\mu J_\mu$$



$$J_i J_i$$

- For all  $s$  :

$$\begin{aligned} \mathcal{A} - \frac{1}{2} \eta \mathcal{A}' + \eta^2 \mathcal{B} &= J \\ \partial \cdot \mathcal{A}' - (2 \partial + \eta \partial \cdot) \mathcal{B} &= 0 \\ \varphi'' - 4 \partial \cdot \alpha - \partial \alpha' &= 0 \end{aligned}$$



# External currents : local case

$$\varphi'' - 4 \partial \cdot \alpha - \partial \alpha' = 0$$

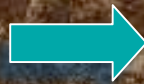
- $\mathcal{K}$  “doubly traceless” using double trace constraint
- $\mathcal{B}$ : determines multiplier  $\beta$  for double trace constraint

$$\mathcal{A} - \frac{1}{2} \eta \mathcal{A}' = J - \eta^2 \mathcal{B} \equiv \mathcal{K}$$

$$\varphi'' - 4 \partial \cdot \alpha - \partial \alpha' = 0$$

$$\text{Traceless Proj} : \sum_0^N \rho_n(D, s) \eta^n V^{[n]}$$

$$\rho_{n+1}(D, s) = - \frac{\rho_n(D, s)}{D + 2(s - n - 2)}$$



$$\mathcal{K} = J + \sum_{n=2}^N \sigma_n \eta^n J^{[n]}$$

$$\sigma_n + [D + 2(s - n - 3)] \{2\sigma_{n+1} + [D + 2(s - n - 4)] \sigma_{n+2}\} = 0$$

$$\sigma_n = (-n + 1) \rho_n(D - 2, s)$$

$$\mathcal{A} = \mathcal{K} + \rho_1(D - 2, s) \eta \mathcal{K}' = \sum_n \rho_n(D - 2, s) \eta^n J^{[n]}$$

$$\sum_{n=0}^N \rho_n(D - 2, s) \frac{s!}{n! (s - 2n)! 2^n} J^{[n]} \cdot J^{[n]}$$

The exchange involves, correctly, a *traceless conserved current*

# External currents : non-local case

*How about the non-local version of the theory?*

*Apparently: different choices for the field equation, EQUIVALENT without currents*

$$S=3: \quad \begin{aligned} \mathcal{F}_{\mu\nu\rho} - \frac{1}{3} \frac{1}{\square} (\partial_\mu \partial_\nu \mathcal{F}'_\rho + \dots) &= 0 \\ \mathcal{F}_{\mu\nu\rho} - \frac{\partial_\mu \partial_\nu \partial_\rho}{\square^2} \partial \cdot \mathcal{F}' &= 0 \end{aligned} \quad \longrightarrow \quad \frac{1}{\square} \eta_{\alpha\beta} \partial_\mu \mathcal{R}^{\mu\alpha\beta}_{;\nu_1\nu_2\nu_3} = 0$$

$$\mathcal{F} = 3 \partial^3 \alpha \quad \longrightarrow \quad \mathcal{F}^{(n)} = (2n + 1) \frac{\partial^{2n+1}}{\square^{n-1}} \alpha^{[n-1]}$$

$$\begin{aligned} \mathcal{F}^{(n+1)} &= \mathcal{F}^{(n)} + \frac{1}{(n+1)(2n+1)} \frac{\partial^2}{\square} \mathcal{F}^{(n)'} - \frac{1}{n+1} \frac{\partial}{\square} \partial \cdot \mathcal{F}^{(n)} \\ \delta \mathcal{F}^{(n)} &= (2n+1) \frac{\partial^{2n+1}}{\square^{n-1}} \Lambda^{[n]} \\ \partial \cdot \mathcal{F}^{(n)} - \frac{1}{2n} \partial \mathcal{F}^{(n)'} &= - \left(1 + \frac{1}{2n}\right) \frac{\partial^{2n+1}}{\square^{n-1}} \varphi^{(n+1)} \end{aligned}$$

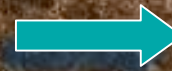
*Bianchi identity: changes after every iteration*



# External currents : non-local case

Naively:

$$\mathcal{G}^{(n)} \equiv \sum_{p=0}^n (-1)^p \frac{(n-p)!}{2^p n!} \eta^p \mathcal{F}^{(n)}[p] = \mathcal{J}$$



$$\mathcal{F}^{(n)} = \sum_{n=0}^n \rho_n(D - 2n, s) \eta^n \mathcal{J}^{[n]}$$

Incorrect current exchange !

**Solution:** modify the non-local Lagrangian equation

$$\mathcal{A}_{nl} = \mathcal{F} - 3\partial^3 \alpha_{nl}; \quad \mathcal{C}_{nl} \equiv \varphi'' - 4\partial \cdot \alpha_{nl} - \partial \alpha'_{nl} = 0$$

$$\partial \cdot \mathcal{A}_{nl} - \frac{1}{2} \partial \mathcal{A}'_{nl} = 0 \quad (\text{Mod } \mathcal{C}_{nl}); \quad \partial \cdot \mathcal{A}_{nl} = 2\mathcal{D}_{nl}$$

$$\mathcal{G}_{nl} = \mathcal{A}_{nl} - \frac{1}{2} \eta \mathcal{A}'_{nl} + \eta^2 \mathcal{D}_{nl} + \dots + \eta^{n+2} \frac{\mathcal{D}_{nl}^{[n]}}{2^n n!} \quad (s = 2n + 4 \text{ or } s = 2n + 5)$$

$$\begin{aligned} \mathcal{A}_{nl} &= \sum_{k=0}^{n+1} a_k \frac{\partial^{2k}}{\square^k} \mathcal{F}^{(n+1)}[k]; \quad a_k = (-1)^{k+1} (2k-1) \frac{n+2}{n-1} \prod_{j=-1}^{k-1} \frac{n+j}{n-j+1} \\ \mathcal{D}_{nl} &= \frac{1}{2} \sum_{k=2}^{n+1} a_k \left\{ \frac{1}{2k-3} \frac{\partial^{2(k-2)}}{\square^{k-2}} \mathcal{F}^{(n+1)}[k] + \frac{2n+4k+1}{2(2k-1)(n-k+1)} \frac{\partial^{2(k-1)}}{\square^{k-1}} \mathcal{F}^{(n+1)}[k+1] \right. \\ &\quad \left. + \frac{n+k+1}{2(n-k)(n-k+1)} \frac{\partial^{2k}}{\square^k} \mathcal{F}^{(n+1)}[k+2] \right\} \end{aligned}$$

For instance :

$$\begin{aligned} \mathcal{A}_3 &= \frac{1}{\square} \partial \cdot \mathcal{R}' - \frac{\partial^2}{2\square^2} \partial \cdot \mathcal{R}'' \\ \mathcal{A}_4 &= \frac{1}{\square} \mathcal{R}'' + \frac{1}{2\square^2} \partial^2 \mathcal{R}''' - 3 \frac{\partial^4}{\square^3} \mathcal{R}^{[4]} \\ \mathcal{A}_5 &= \frac{1}{\square^2} \partial \cdot \mathcal{R}'' + \frac{2}{3\square^3} \partial^2 \partial \cdot \mathcal{R}''' - 3 \frac{\partial^4}{\square^4} \partial \cdot \mathcal{R}^{[4]} \end{aligned}$$

$$\begin{aligned} \mathcal{D}_4 &= -\frac{3}{8} \frac{1}{\square} \mathcal{R}^{[4]} \\ \mathcal{D}_5 &= -\frac{5}{8\square^2} \partial \cdot \mathcal{R}^{[4]} \end{aligned}$$

# VD-V-Z Discontinuity for HS



$$m = 0 : T_{\mu\nu} T^{\mu\nu} - \frac{1}{2} (T')^2 \quad (\text{van Dam, Veltman, Zakharov, 1970})$$

$$m \neq 0 : T_{\mu\nu} T^{\mu\nu} - \frac{1}{3} (T')^2$$

For all  $s$  and  $D$ ,  $m=0$  :

$$\sum_{n=0}^N \rho_n(D-2, s) \frac{s!}{n! (s-2n)! 2^n} J^{[n]} \cdot J^{[n]}$$

$$\rho_{n+1}(D, s) = - \frac{\rho_n(D, s)}{D+2(s-n-2)}$$

$$s = 2 : T_{\mu\nu} T^{\mu\nu} - \frac{1}{D-2} (T')^2$$

- *VDVZ discontinuity* follows in general comparing  $D$  and  $(D+1)$  massless exchanges
- First present for  $s=2$  via  $D$ -dependence of  $\rho(D, s)$
- *For all  $s$* : can describe irreducibly a massive field a' la Scherk-Schwarz from  $(D+1)$  dimensions :  
 [ e.g. for  $s=2$  :  $\hat{h}_{\mathcal{MN}} \rightarrow (\hat{h}_{\mu\nu} \cos(my), \mathcal{A}_\mu \sin(my), \varphi \cos(my))$  ]

$$\sum_{n=0}^N \rho_n(D-2, s) \frac{s!}{n! (s-2n)! 2^n} J^{[n]} \cdot J^{[n]}$$

$$\sum_{n=0}^N \rho_n(D-1, s) \frac{s!}{n! (s-2n)! 2^n} J^{[n]} \cdot J^{[n]}$$

(A)dS extension, first discussed, for  $s=2$ , by Higuchi and Porrati  
 Discontinuity  $\rightarrow$  smooth interpolation in  $(mL)^2$



# Outlook

- *Precise link with String Theory :*
  - *Vasiliev equations : effective theory for first open Regge trajectory ?*
- *Systematics of HS currents and interactions :*
  - *Where does String Theory lie within HS gauge theory ?*
- *HS geometry (vs algebraic structure as in Vasiliev equations) :*
  - *Key lesson for String Theory : background independence*